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1981 J. Phys. A: Math. Gen. 14 2917

(http://iopscience.iop.org/0305-4470/14/11/015)

J. Phys. A: Math. Gen. **14** (1981) 2917–2925. Printed in Great Britain

'Proper time' and the Dirac equation

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Received 4 December 1980, in final form 14 April 1981

Abstract. In the theory of the Dirac equation in quantum mechanics, the Dirac matrix β is taken to connect 'proper time' and coordinate time rates of change of operators. The usual Heisenberg equation for the coordinate time rate of change of an operator then leads to a Heisenberg-like equation for its proper time rate (the 'Hamiltonian' being analogous to the generator of proper time translations in classical mechanics). Certain terms have to be set to zero by the use of the Dirac equation, and this is here justified. The equivalence of the formalisms under these conditions is shown and the method entitles us to derive some well known results for the Dirac equation in a comparatively effortless manner. The present work justifies methods introduced by H C Corben.

1. Introduction

Corben (1968) draws some detailed comparisons between the helical solutions of the classical equations

$$\dot{p}^{\mu}=0, \qquad \dot{s}^{\mu\nu}+2p^{[\mu}\dot{x}^{\nu]}=0, \qquad s^{\mu\nu}\dot{x}_{\nu}=0 \tag{1.1}$$

(where a dot denotes differentiation with respect to the proper time τ) for a free particle with spin in classical mechanics, and the phenomenon of 'zitterbewegung' of the Dirac equation in quantum mechanics (pp 185–90—where some theorems are employed; see also Corben (1961)). For the purposes of illustrating these comparisons Corben defines and uses an operator in quantum mechanics which gives rise to the 'proper time' differentiation of operators according to the Heisenberg equation

$$\mathrm{d}X/\mathrm{d}\tau=(1/\mathrm{i}\hbar c)[X, \mathscr{H}]. \tag{1.2a}$$

(We shall follow Corben by referring to $\mathrm{d}X/\mathrm{d}\tau$ as the 'proper time derivative' of X.) For the Dirac equation, $\mathscr{H}$ is taken to be $-\gamma^{\mu}\hat{p}_{\mu}$ (in the absence of interaction). The use of this operator, however, appears to be imprecise. The following work is an attempt to show how the use of such an operator can be justified and can lead to consistent results. We shall do this (a) by actually calculating the operator $\mathscr{H}$ from the Hamiltonian $H=\beta mc^2+c(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})$ for the Dirac equation (where it will be seen that certain terms have to be dropped in accordance with the Dirac equation—this calculation appears not to have been carried out by Corben), and (b) by using it to obtain specific well known results according to certain rules which we can 'legitimise'. The use of the operator $\mathscr{H}$ to

† This work was carried out in part while on leave of absence between October 1979 and September 1980.

0305-4470/81/112917+09\$01.50

obtain such results appears to be a comparatively effortless one, which can be justified, and there are certain advantages which are parallelled in other situations (see below).

In the following work, the Dirac matrix β is assumed to connect proper and coordinate time derivatives of operators in the theory of the Dirac equation in quantum mechanics. For any operator X in the Heisenberg representation, we define

$$\frac{\mathrm{d}X}{\mathrm{d}\tau} \stackrel{\text{def}}{=} \frac{\beta}{c}\frac{\mathrm{d}X}{\mathrm{d}t}, \tag{1.2b}$$

where $\mathrm{d}X/\mathrm{d}t$ is given by the ordinary Heisenberg equation of motion. This definition of proper time differentiation helps to simplify calculations where coordinate time derivatives of operators are required. The left-hand side of (1.2b), at this stage, is merely intended to be alternative notation for the right, there being no sense of actual differentiation of the operator with respect to the proper time of special relativity except in so far as we shall here show that the right-hand sides of (1.2b) and (1.2a) are connected, so that we have a Heisenberg-like equation. Granted that we can show this, there is some justification in referring to the left-hand side as a 'derivative' with respect to a time quantity (which has the dimensions of length), and the notion of calling it the 'proper time derivative of X' arises by classical analogy, since if one discusses the classical Lagrangian and Hamiltonian mechanics of a single particle in Lorentz space, with the proper time τ rather than the coordinate time t as an independent variable, using a Poisson bracket based on the four quantities $x^{\mu} = (ct, \boldsymbol{r})$ and their conjugate momenta p_{μ}, equations of motion very similar to (1.2a) are obtained†. (The classical counterpart of $\mathscr{H}$ is not the total energy and may even be a function of the coordinates and momenta which is numerically equal to zero (Mann 1974, pp 127–31). There is also a complication arising in the existence of the velocity constraint $\dot{x}^{\mu}\dot{x}_{\mu} = 1$ for a free particle, and this can be dealt with by using Lagrange multipliers and the classical Hamiltonian theory invented by Dirac (1964) for a degenerate Lagrangian which uses 'modified' Poisson brackets in place of Poisson brackets, but the same considerations referring to (1.2a) apply.)

The use of the definition (1.2b) of 'proper time differentiation' of operators in quantum mechanics involving the Dirac matrix β contributes to a certain 'economy' in calculations where derivatives of operators are required which is not found when using coordinate time derivatives. Although no actual differentiation with respect to τ is implied, it appears that the situation has a kind of parallel in the situation met in electromagnetism, where the fields of a non-uniformly moving charge are calculated in terms of proper time derivatives, for example. The expressions obtained are more concise than those calculated directly by using coordinate time derivatives. We assume a dependence of the charge's path upon proper time rather than upon coordinate time, and the proper time derivatives arising in the four-acceleration in these concise

† The equations of motion in terms of this Poisson bracket are

$$\mathrm{d}g/\mathrm{d}\tau = (\partial\mathscr{H}/\partial x^{\mu})\,\partial g/\partial p_{\mu} - (\partial\mathscr{H}/\partial p_{\mu})\,\partial g/\partial x^{\mu},$$

where $p_{\mu} = -\partial L/\partial\dot{x}^{\mu}$. When written for a g which is explicitly time dependent, say $g = g(\boldsymbol{r}, \boldsymbol{p}, t)$, these equations are seen to include the 'explicit' term $\partial g/\partial t$ arising in the usual equations of motion based on the three-dimensional Poisson bracket:

$$\mathrm{d}g/\mathrm{d}\tau = (\beta/c)(\partial g/\partial t + \{g, H\}_{\text{ord.PB}}), \qquad \beta = c\,\mathrm{d}t/\mathrm{d}\tau\ (= -\partial\mathscr{H}/\partial p_0).$$

(The component p^0 represents the energy of the particle.)

expressions are easily related to coordinate time derivatives by the use of the differential relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}=\frac{\beta}{c}\frac{\mathrm{d}}{\mathrm{d}t'}, \qquad \beta=c\frac{\mathrm{d}t'}{\mathrm{d}\tau}=\left(1-\frac{v^2}{c^2}\right)^{-1/2};$$

and then the usual complicated three-vector expressions for the retarded electromagnetic fields may be obtained, where t' is the retarded time (see, for example, Møller 1952, p 150). In contrast, the 'proper time derivative' of operators in quantum mechanics appears to have had little use.

It is worth remarking that we have been unable to find any reference to the proper time derivative of operators in standard works on quantum mechanics. We find, however, that one text (Bethe 1964, p 207) comes close to the present work in the non-contentious statement: 'No physical interpretation is given to the β matrix but the following relations can be verified:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}+\frac{\hbar\mathrm{i}}{2mc}\beta\boldsymbol{\alpha}\right)=\frac{\beta\hat{\boldsymbol{p}}}{m}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(t+\frac{\hbar\mathrm{i}}{2mc^2}\beta\right)=\frac{\beta}{mc^2}H \qquad \text{(etc).} \tag{1.3}$$

The significance of these results is not understood.' The proper time derivative may have some bearing on these results.

2. Definitions

We use the definition (1.2b) and employ the following notation,

$$\dot{X}\equiv\frac{\mathrm{d}X}{\mathrm{d}\tau}\overset{\mathrm{def}}{=}\frac{\beta}{c}\frac{\mathrm{d}X}{\mathrm{d}t}, \tag{2.1}$$

and henceforth denote by a dot the so-called 'proper time rate of change' of an operator. (We have already stated that this does not necessarily imply actual differentiation of the operator with respect to τ in quantum mechanics.) For the coordinate time rate of change, $\mathrm{d}X/\mathrm{d}t$, we employ the Heisenberg equation

$$\frac{\mathrm{d}X}{\mathrm{d}t}\overset{\mathrm{def}}{=}\frac{\partial X}{\partial t}+\frac{1}{\mathrm{i}\hbar}[X,H] \tag{2.2}$$

where $H=\beta mc^2+c(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})$ is the Hamiltonian operator for the Dirac equation. The first term of (2.2) is included for the sake of operators that depend on t explicitly, and is indispensible for the interchange of (2.1) and (1.2a) in general.

The normal interpretation of (2.2) is by the use of matrix elements between time-dependent Dirac basis wavefunctions, and the method of its derivation (such as the one given by Dicke and Wittke (1960, pp 181–2) for example) then allows us to represent the first term of (2.2) as follows:

$$\partial X/\partial t=(1/\mathrm{i}\hbar)(\hat{E}X-X\hat{E}), \tag{2.3}$$

where the energy operator

$$\hat{E}\overset{\mathrm{def}}{=}\mathrm{i}\hbar\,\partial/\partial t$$

is the first component of the four-vector operator

$$\hat{p}_\mu=\mathrm{i}\hbar c\,\vec{\partial}_\mu=(\mathrm{i}\hbar\,\partial/\partial t,\mathrm{i}\hbar c\,\boldsymbol{\nabla})=(\hat{E},-c\hat{\boldsymbol{p}}), \tag{2.4}$$

which acts on all quantities to the right. (We have introduced circumflex accents, as used, for example, by Aitchison (1972), to denote operators. The usual notation for $\hat{p}^{\mu}$ has been multiplied by c. The notation without circumflex accents denotes eigenvalues of the corresponding operators.)

We can express the proper time rate of change of the operator X directly from (2.1), (2.2) and (2.3), remembering that operator equations are interpreted by the use of matrix elements:

$$\begin{aligned} \mathrm{i}\hbar c\dot{X} &\doteq \mathrm{i}\hbar\beta\, \mathrm{d}X/\mathrm{d}t \\ &= \beta[\hat{E}, X] + \beta[X, H] = \beta[X, H - \hat{E}] \\ &= [X, \beta(H - \hat{E})] + [\beta, X](H - \hat{E}). \end{aligned}$$

The operator $\beta\hat{E}$ is one of the terms of $\gamma^{\mu}\hat{p}_{\mu}$ where the standard representation $\gamma^{\mu} = (\beta, \beta\boldsymbol{\alpha})$ is used. In the case of no interaction, where the Hamiltonian for the Dirac equation is the usual one, this equation reads†

$$c\dot{X} = (1/\mathrm{i}\hbar)[X, \mathscr{H}] + (1/\mathrm{i}\hbar)[\beta, X](H - \hat{E}) \tag{2.5}$$

with $\mathscr{H} = -\gamma^{\mu}\hat{p}_{\mu}$. We may apply the considerations that led to this equation to the second proper time derivative of X. We replace the operator X in (2.5) (which is assumed to be evaluated as matrix elements between wavefunctions) by the operator representative for $c\dot{X}$ itself, i.e. by the same right-hand side of (2.5), and we obtain the equivalent operator expression for $c^2\ddot{X}$:

$$\begin{aligned} c^2\ddot{X} &= \beta(\mathrm{d}/\mathrm{d}t)(\beta\, \mathrm{d}X/\mathrm{d}t) \\ &= \frac{1}{\mathrm{i}\hbar}\left[\frac{1}{\mathrm{i}\hbar}[X, \mathscr{H}], \mathscr{H}\right] + \frac{1}{\mathrm{i}\hbar}\left[\frac{1}{\mathrm{i}\hbar}[\beta, X](H - \hat{E}), \mathscr{H}\right] \\ &\quad + \frac{1}{\mathrm{i}\hbar}\left[\beta, \frac{1}{\mathrm{i}\hbar}[X, \mathscr{H}]\right](H - \hat{E}) + \frac{1}{\mathrm{i}\hbar}\left[\beta, \frac{1}{\mathrm{i}\hbar}[\beta, X](H - \hat{E})\right](H - \hat{E}). \end{aligned} \tag{2.6}$$

The circumstances under which the second term of (2.5) may be omitted are where X commutes with β; but quite generally, since the normal interpretation of operator equations like (2.5) is by the use of matrix elements between basis wavefunctions, the second term of (2.5) may be omitted—the operator $H - \hat{E}$, acting on a Dirac basis wavefunction $\psi_n(x)$, vanishes by virtue of the Dirac equation $H\psi_n = \hat{E}\psi_n$. Likewise, the second, third and fourth terms of (2.6) may be omitted. The third and fourth terms vanish by virtue of the Dirac equation and the second term also vanishes by virtue of the Dirac equation because $(H - \hat{E})\mathscr{H}\psi_n = (H - \hat{E})(-mc^2\psi_n) = 0$. The omission of these terms from (2.5) and (2.6) merely produces a different expression which has the *same* interpretation by the use of matrix elements between wavefunctions and which is more convenient to use.

Definition. When an operator is legitimately simplified by operating the extreme RH factors of the operator (or the extreme LH ones) on the Dirac basis wavefunctions in the

† With regard to the logic of the equation preceding (2.5), note that β, H and $\hat{E}$ are all Hermitian operators, but that the product $\beta(H - \hat{E}) \equiv mc^2 I - \gamma^{\mu}\hat{p}_{\mu}$ is *not* Hermitian. Consequently, if $(H - \hat{E})|\text{any state}\rangle = 0$ this does not imply $[X, \beta(H - \hat{E})] = 0$ or $\dot{X} = 0$. Note also that if X is a Hermitian operator $\mathrm{d}X/\mathrm{d}t$ is also Hermitian from (2.2) because the separate terms are. This fact is again verifiable from (2.5) on taking the Hermitian conjugate of $\beta\dot{X}$. We find $(\dot{X})^{\dagger} = \beta\dot{X}\beta$.

matrix elements of the operator, the notation $\doteq$ will be used to indicate this simplification.

Thus, under the circumstances described above, we have derived

$$c\dot{X} \doteq (1/\mathrm{i}\hbar)[X, \mathscr{H}], \qquad c^2\ddot{X} \doteq (1/\mathrm{i}\hbar)[(1/\mathrm{i}\hbar)[X, \mathscr{H}], \mathscr{H}] \tag{2.7}$$

with

$$\mathscr{H} = -\gamma^\mu \hat{p}_\mu. \tag{2.8}$$

3. Some uses of the proper time derivative

For the spin-$\frac{1}{2}$ particle the total angular momentum in a certain state is the expectation value of the operator

$$\boldsymbol{J} = \boldsymbol{r} \wedge \hat{\boldsymbol{p}} + \tfrac{1}{2}\hbar\boldsymbol{\sigma}, \tag{3.1}$$

i.e.

$$\int \psi^\dagger \boldsymbol{J} \psi \, \mathrm{d}\boldsymbol{x}. \tag{3.2}$$

This quantity is a constant of the motion, whereas the expectations of the orbital and spin parts separately are not constants. These ideas are extended to four dimensions by defining

$$j^{\mu\nu} = -2x^{[\mu}\hat{p}^{\nu]} + s^{\mu\nu} \tag{3.3}$$

where

$$s^{\mu\nu} = -\tfrac{1}{2}\hbar c \sigma^{\mu\nu} \qquad (\sigma^{\mu\nu} = \tfrac{1}{2}\mathrm{i}[\gamma^\mu, \gamma^\nu]). \tag{3.4}$$

The square brackets in (3.3) denote antisymmetrisation with the order maintained, e.g.

$$2a^{[\mu}b^{\nu]} = a^\mu b^\nu - a^\nu b^\mu,$$

and the space–space and space–time parts of (3.3) are as follows:

$$(j^{23}, j^{31}, j^{12}) = -c(\boldsymbol{r} \wedge \hat{\boldsymbol{p}} + \tfrac{1}{2}\hbar\boldsymbol{\sigma}), \tag{3.5}$$

$$(j^{01}, j^{02}, j^{03}) = \boldsymbol{r}\hat{E} - c^2 t\hat{\boldsymbol{p}} - \tfrac{1}{2}\mathrm{i}\hbar c\boldsymbol{\alpha}. \tag{3.6}$$

We have suppressed the unit matrix in the right-hand sides of these expressions. With regard to the non-Hermiticity of the components of (3.6), see footnote to equation (3.9). We now have t occurring explicitly in (3.6), and the constancy of all components (3.3) can easily be checked by means of our formulae (2.7), which were derived on the basis of the Heisenberg equation containing the extra term $\partial X/\partial t$. We first derive the simplified expression for $\dot{s}^{\mu\nu}$:

$$\begin{aligned} \dot{s}^{\mu\nu} &\doteq (1/\mathrm{i}\hbar c)[s^{\mu\nu}, -\gamma^\sigma \hat{p}_\sigma] \\ &= \tfrac{1}{4}[2\gamma^\mu\gamma^\nu - 2g^{\mu\nu}I, \gamma^\sigma]\hat{p}_\sigma = \tfrac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\mu\gamma^\nu)\hat{p}_\sigma \\ &= \tfrac{1}{2}(\gamma^\mu\gamma^\nu\gamma^\sigma - 2g^{\sigma\mu}\gamma^\nu + 2\gamma^\mu g^{\sigma\nu} - \gamma^\mu\gamma^\nu\gamma^\sigma)\hat{p}_\sigma \\ &= -2\hat{p}^{[\mu}\gamma^{\nu]}. \end{aligned} \tag{3.7}$$

Hence from (3.3) and (3.7) we obtain the proper (and hence the coordinate) time rate of $j^{\mu\nu}$ vanishing by virtue of the Dirac equation:

$$\begin{aligned} \dot{j}^{\mu\nu} &\underset{\cdot}{=} -(2/\mathrm{i}\hbar c)[x^{[\mu}\hat{p}^{\nu]}, -\gamma^{\sigma}\hat{p}_{\sigma}] - 2\hat{p}^{[\mu}\gamma^{\nu]} \\ &= -(2/\mathrm{i}\hbar c)[x^{[\mu}, -\gamma^{\sigma}\hat{p}_{\sigma}]\hat{p}^{\nu]} - 2\hat{p}^{[\mu}\gamma^{\nu]} \\ &= -2\gamma^{\sigma}\delta^{[\mu}{}_{\sigma}\hat{p}^{\nu]} - 2\hat{p}^{[\mu}\gamma^{\nu]} \\ &= 0. \end{aligned} \tag{3.8}$$

This result includes the usual result for the space–space components only. The latter result holds without the use of the Dirac equation, but the result for the space–time components requires it.

Hilgevoord and Wouthuysen (1963) performed a different splitting of this total angular momentum tensor in such a way that both parts were separately constant. The space–space parts of their new spin angular momentum were related to the so-called mean spin operator of the Foldy–Wouthuysen theory (1950). Their method was based on the fact that every solution of the Dirac equation is also a solution of the Klein–Gordon equation. Starting from a Lagrangian for the Klein–Gordon equation of a spinor function, with the Dirac equation introduced later as a subsidiary condition on the solutions, they were enabled to obtain the new constant spin angular momentum. That the new total angular momentum thus obtained was no different from the usual total angular momentum of the Dirac theory, was checked independently by defining the invariant inner product for two spinor solutions of the Klein–Gordon equation and by making the spinor solutions also satisfy the Dirac equation, so that the invariant inner product for the Klein–Gordon equation reduced to the well known one in the Dirac theory. The conserved quantities in the Dirac theory thus arose as Klein–Gordon expectation values. Their method was a general one and enabled them to obtain virtually all the conserved quantities in the Dirac theory.

We shall consider only the new spin angular momentum which they obtained†:

$$S^{\mu\nu} \overset{\text{def}}{=} s^{\mu\nu} + (\mathrm{i}\hbar/mc)\gamma^{[\mu}\hat{p}^{\nu]}. \tag{3.9}$$

It is an elementary exercise to show, using our formulae (2.7), that (3.9) is constant:

$$\begin{aligned} \dot{S}^{\mu\nu} &\underset{\cdot}{=} -2\hat{p}^{[\mu}\gamma^{\nu]} + (1/mc^2)[\gamma^{[\mu}\hat{p}^{\nu]}, -\gamma^{\sigma}\hat{p}_{\sigma}], \qquad \text{using (3.7)}, \\ &= -2\hat{p}^{[\mu}\gamma^{\nu]} - (1/mc^2)[\gamma^{[\mu}, \gamma^{\sigma}]\hat{p}_{\sigma}\hat{p}^{\nu]} \\ &= -2\hat{p}^{[\mu}\gamma^{\nu]} - (1/mc^2)(2\gamma^{[\mu}\gamma^{\sigma} - 2g^{[\mu\sigma}I)\hat{p}_{\sigma}\hat{p}^{\nu]} \\ &\underset{\cdot}{=} -2\hat{p}^{[\mu}\gamma^{\nu]} - (1/mc^2)(2\gamma^{[\mu}\hat{p}^{\nu]}mc^2) \\ &= 0. \end{aligned} \tag{3.10}$$

† Note that in this operator, $\hat{p}^0 = \hat{E} = \mathrm{i}\hbar\partial/\partial t$, but that in a similar operator quoted by de Groot and Suttorp (1972, p 423, formula (87)), who refer to Hilgevoord and Wouthuysen's paper, $\hat{p}^0 = H = \beta mc^2 + c(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}})$ is taken. Consequently, at first sight, only the space–space components of the similar operator of de Groot and Suttorp are constant (which they mention). (Since the substitution $\hat{p}^0 = H$ in (3.9) is a legitimate simplification of the matrix elements of the operator by the Dirac equation which does not affect the proper time rate of change of the operator, both the space–time components of this Hermitian operator and the Hermitian operator resulting from (3.3) by this substitution are also constant.) The spin tensor (3.9) was also obtained, slightly earlier than Hilgevoord and Wouthuysen's derivation, by Fradkin and Good (1961) in a completely different manner. The space–space components only were derived many years before, by Pryce (1948).

We conclude this discussion by justifying a theorem of Corben which produces (apart from other results) another operator, equivalent to (3.9) in the sense of $\doteqdot$ and also constant, which could replace it. This theorem of Corben makes use of *two* proper time differentiations, based on two quite separate 'Hamiltonians'. This part of his theorem we have not been able to substantiate in the present formalism. We state and prove below a revised theorem.

Theorem. If X *is any operator that commutes with* $\hat{p}^{\mu}\hat{p}_{\mu}$ $(\equiv -\hbar^2c^2\Box^2)$, *then the operator*

$$X' = X + (\mathrm{i}\hbar/2mc)\dot{X} \tag{3.11}$$

(or any $\doteqdot$ *equivalent of it) is constant.*

From (2.7) we have

$$\begin{aligned} X' &\doteqdot X + (1/2mc^2)[X, \mathscr{H}] \\ &\doteqdot X + (1/2mc^2)[X, \mathscr{H}] - (1/mc^2)X(mc^2I + \mathscr{H}) \\ &= -(1/2mc^2)(X\mathscr{H} + \mathscr{H}X). \end{aligned} \tag{3.12}$$

We have stated previously that any simplification to the right-hand factors of an operator by the Dirac equation does not affect the proper time rate of that operator. Consequently, we may use (3.12) in place of X' when evaluating the proper time derivative of X':

$$\begin{aligned} \dot{X}' &\doteqdot (1/\mathrm{i}\hbar c)[X', \mathscr{H}] \\ &\doteqdot (\mathrm{i}/2m\hbar c^3)[X\mathscr{H} + \mathscr{H}X, \mathscr{H}] \\ &= (1/\mathrm{i}\hbar c)[X, -(1/2mc^2)\mathscr{H}^2]. \end{aligned}$$

Since $\mathscr{H}^2 = (\gamma^{\mu}\hat{p}_{\mu})(\gamma^{\nu}\hat{p}_{\nu}) = \hat{p}^{\mu}\hat{p}_{\mu}$, the result is proved.

Inserting $s^{\mu\nu}$ in (3.11) produces the constant spin tensor (3.9) of Hilgevoord and Wouthuysen:

$$\begin{aligned} s'^{\mu\nu} &= s^{\mu\nu} + (\mathrm{i}\hbar/2mc)\dot{s}^{\mu\nu} \\ &\doteqdot s^{\mu\nu} + (\mathrm{i}\hbar/mc)\gamma^{[\mu}\hat{p}^{\nu]}, \qquad \text{by (3.7).} \end{aligned}$$

The variety of conserved tensors found by these authors for the free Dirac theory, are re-found by inserting the matrices I, γ^{μ}, $\sigma^{\mu\nu}$, $\mathrm{i}\gamma^5\gamma^{\mu}$ and γ^5. The first and last matrices merely give the unit matrix I and the zero matrix, respectively; the second gives $\hat{p}^{\mu}/mc^2$; the third has been dealt with under $s^{\mu\nu}$; and the fourth gives the four-vector operator of Bargmann and Wigner in either of the equivalent (in our terminology, $\doteqdot$ equivalent) forms used by Fradkin and Good (1961):

$$\mathrm{i}\gamma^5(\gamma^{\mu} - \hat{p}^{\mu}/mc^2), \qquad \tfrac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\alpha\beta}\gamma_{\nu}\gamma_{\alpha}\hat{p}_{\beta}/mc^2$$

(more precisely, it gives further $\doteqdot$ equivalent versions of these since $\hat{p}^0 = H$ is taken by these authors). Any constant combination of the sixteen linearly independent Dirac matrices produces a Dirac constant in this way.

Finally, we indicate how the idea of eliminating the zitterbewegung may be introduced using this terminology† (cf Corben 1968, p 190). Referring to the analogous helical solutions of the classical equations (1.1) (whose operator counterparts are $\dot{\hat{p}}^{\mu} = 0$, (3.7), and the identity $\sigma^{\mu\nu}\gamma_\nu + \gamma_\nu\sigma^{\mu\nu} \equiv 0$), the classical transformation

$$\bar{x}^{\mu} = x^{\mu} + k\ddot{x}^{\mu},$$

for suitable k, orthogonally projects a typical point on the helical path onto the helical axis (having direction p^{μ}). Making a similar transformation for operators, we have

$$\bar{x}^{\mu} \mathrel{\overset{.}{=}} x^{\mu} + k(2/\hbar c)\sigma^{\mu\nu}\hat{p}_\nu,$$

where $\dot{x}^{\mu} = \gamma^{\mu}$. The value of k chosen to eliminate the zitterbewegung is $k = (\hbar/2mc)^2$ whence $\dot{\bar{x}}^{\mu} \mathrel{\underset{.}{=}} I\hat{p}^{\mu}/mc^2$. With this it can easily be shown that the new spin angular momentum tensor $j^{\mu\nu} + 2\bar{x}^{[\mu}\hat{p}^{\nu]}$ is $\mathrel{\underset{.}{=}}$ equivalent to that of Hilgevoord and Wouthuysen, and the new orbital and spin values are constants.

4. Conclusion

We have attempted to give a logical basis for the representation of the generator of proper time translations in quantum mechanics. The idea for this is not entirely new, but we have been unable to find any reference to it in well known texts in quantum mechanics. Previous proper time formalisms, essentially different from that described here, have been given by Nambu (1950) and by Szamosi (1961). Our attempt has been founded on linking proper time and coordinate time derivatives of operators via the Dirac matrix β, following Corben (1968), and we have evolved a definite process in which the right-hand factors of operators are simplified by using the Dirac equation only after commutators of these operators have been evaluated, which agrees with the normal meaning of operators in terms of matrix elements between wavefunctions. We have demonstrated the use of the new methods by obtaining the operators of Hilgevoord and Wouthuysén (1963) (and those of Fradkin and Good (1961)), showing their constancy using these techniques.

Acknowledgment

The author wishes to express his thanks to the referees for their detailed advice concerning the presentation of this paper.

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† We have been unable to apply the methods described here to the Foldy–Wouthuysen transformation, but note in passing that their mean position operator $\boldsymbol{X} = S_{\mathrm{FW}}^{-1}\boldsymbol{x}S_{\mathrm{FW}}$ (1950, formula (23), also reproduced by Schweber (1961, p 94) and by Margenau and Murphy (1964, p 537)) is incorrect in the third term. We have verified by direct calculation that a revised expression correctly satisfies $\mathrm{d}\boldsymbol{X}/\mathrm{d}t = c^2\hat{\boldsymbol{p}}H/E_p^2$, $\boldsymbol{X} \wedge \boldsymbol{X} = \boldsymbol{0}$, but that the unrevised expression does not satisfy the first of these and probably also does not satisfy the second.

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